Chapter 2 First order Partial differential equations

$$f(x, y, u, u_x, u_y) = 0$$ "In two variable"

① Linear first order PDE

$$a(x,y)\, u_x + b(x,y)\, u_y + C(x,y)\, u = d(x,y)$$

② Semi-Linear first order PDE

$$a(x,y)\, u_x + b(x,y)\, u_y = C(x,y,u)$$

The function $C(x,y,u)$ is non-Linear function in u

$$a(x,y)\, u_x + b(x,y)\, u_y - C(x,y,u)\, u = 0$$

وجود الـ u

أصبح $C(x,y,u)$ Non-Linear

$C(x,y,u)$ is a function in u

③ Almost-Linear first order PDE
"NoT-existed"

④ Quasi-Linear first order PDE

$$a(x,y,u)\, u_x + b(x,y,u)\, u_y - C(x,y,u)\, u = 0$$

⑤ Non-Linear first order PDE → هذه الطريقة ليست مهمة لنا الآن

$$a(x,y,u_x,u_y)\, u_x + b(x,y,u_x,u_y)\, u_y - C(x,y,u)\, u = 0$$

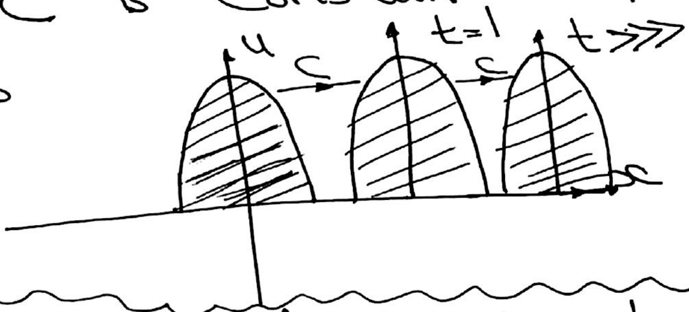
ex → $u_x u_y = 1$
→ $u_x^2 + u = 1$

$u_t + C u_x = 0$ → Celebrated 1^{st} order P.D.E

Trafic flow equation

C is Constant = speed

when $t=0$



Chock wave

Method Ⅰ The Method of characteristics

$$a(x,y,u)\, u_x + b(x,y,u)\, u_y - C(x,y,u) = 0$$

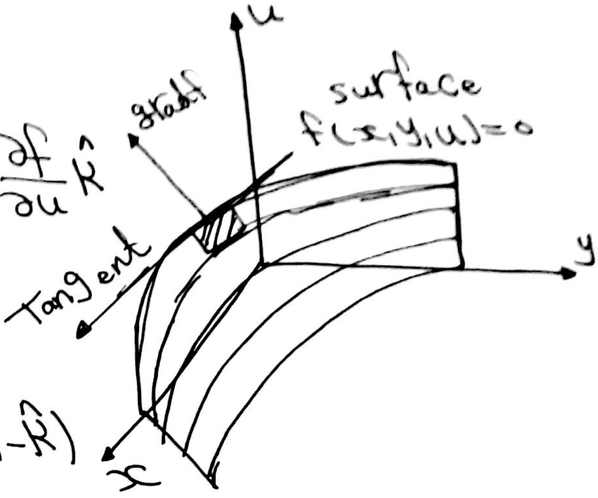
or $a u_x + b u_y - C = 0$ → Quasi-Linear

The sol is surface in xyu space

$$f(x,y,u) = u(x,y) - u = 0$$

$$\text{grad } f = \bar{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial u}\hat{k}$$

$$\text{grad } f = u_x \hat{i} + u_y \hat{j} - 1\hat{k}$$



$$a u_x + b u_y - c = 0$$

$$= (a\hat{i} + b\hat{j} + c\hat{k}) \odot (u_x\hat{i} + u_y\hat{j} - \hat{k})$$

$$= 0$$

المماس
Tangent $\bar{f}$

العمودي
على السطح
Normal $\bar{f}$

المعادلات البارامترية

The Parametric equation of the solution

$$x = x(t), \quad y = y(t), \quad u = u(t)$$

$t \longrightarrow$ Parameter

$$\text{Tangent} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{du}{dt}\right)$$

$$a = \frac{dx}{dt}, \quad b = \frac{dy}{dt}, \quad c = \frac{du}{dt}$$

Eliminate dt $\quad dt = \frac{dx}{a}, \quad dt = \frac{dy}{b}, \quad dt = \frac{du}{c}$

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}$$

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)} \longrightarrow$$ The charactrista equation of The surface

The solution $\phi(x,y,u) = C_1$

$$\psi(x,y,u) = C_2$$

The general solution

$$f(x,y,u) = f(\phi, \psi) = 0$$

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Example 2.5.1

Find the general solution of the first order Linear PDE

$$xu_x + yu_y = u$$

$$xu_x + yu_y - u = 0$$

$$a = x, \quad b = y, \quad -c = -u, \quad c = u$$

Characteristic equation

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}$$

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\ln x = \ln y = c_1$$

$$\ln x - \ln y = c_1 \longrightarrow c_1 = \ln \frac{x}{y} \longrightarrow \frac{x}{y} = e^{c_1} = k$$

$$\phi(x,y,u) = \frac{x}{y}$$

$$\int \frac{dy}{y} = \int \frac{du}{u}$$

$$\therefore \ln y = \ln u + c_2$$

$$c_2 = \ln y - \ln u = \ln\left(\frac{y}{u}\right)$$

$$\psi(x,y,u) = \ln\left(\frac{y}{u}\right) \cancel{-u} = c_2$$

$$f(\phi, \psi) = f\left(\frac{x}{y}, \ln\left(\frac{y}{u}\right) \cancel{-u}\right) = 0$$

$$\frac{x}{y} \cdot \left(\ln\left(\frac{y}{u}\right) \cancel{-u}\right) = 0$$

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Example 2.5.2

$$xu_x + yu_y = nu$$

$$xu_x + yu_y - nu = 0$$

$$au_x + bu_y - c = 0$$

$$a = x, \quad b = y, \quad c = nu$$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{nu}$$

$$\int \frac{dx}{x} = \int \frac{dy}{y} \qquad \ln x = \ln y + c_1$$

$$c_1 = \ln \frac{x}{y} \qquad \therefore \frac{x}{y} = e^{c_1} = k$$

$$\phi = \frac{x}{y}$$

$$\int \frac{dy}{y} = \int \frac{du}{nu}$$

$$\ln y = \frac{1}{n} \ln u + C_2$$

$$C_2 = \ln y - \ln (u)^{1/n}$$

$$= \ln \left(\frac{y}{u^{1/n}} \right)$$

$$\psi = \frac{y}{u^{1/n}}$$

$$f(\phi, \psi) = 0$$

$$f\left(\frac{x}{y}, \frac{y}{u^{1/n}} \right) = 0$$

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$\boxed{3}$ Show that $u = f(x, y)$

f arbitary

$$x u_x - y u_y = 0$$

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{du}{0} \longrightarrow u = \text{constant}$$

$$\ln x = -\ln y + \text{Const}$$

$$\ln x + \ln y = \text{Const}$$

$$\ln (xy) = \text{Const}$$

$$\phi = xy$$

$$f(\phi, \psi) = f(xy) = \text{Const}$$

$$u = f(xy)$$

$$u_x = \frac{\partial f}{\partial x} \cdot y$$

$$u_y = \frac{\partial f}{\partial y} \cdot x$$

another

$$u = f(z), \quad z = xy$$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = y f'$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = x f'$$